CHAPTER 4: PORTFOLIO THEORY

Chapter 4 discusses the theory behind modern portfolio management. Essentially, portfolio managers construct investment portfolios by measuring a portfolio's risks and returns. An understanding of the relationship between portfolio risk and correlation is critical to understanding modern portfolio theory. A grasp of variance, standard deviation, the Markowitz model, the risk-free rate, and the Capital Asset Pricing Model (CAPM) are also needed.

Calculating Portfolio Returns

- The expected return of a portfolio is simply the weighted average of returns on individual assets within the portfolio, weighted by their proportionate share of the portfolio *at the beginning of the measurement*.

Example 1:

Oliver's portfolio holds security A, which returned 12.0% and security B, which returned 15.0%. At the beginning of the year 70% was invested in security A and the remaining 30% was invested in security B. Calculate the return of Oliver's portfolio over the year.

$$\mathbf{R_p = (.6x12\%)+(.3x15\%) = 12.9\%}$$

Example 2:

Oliver's portfolio holds security A, which returned 12.0%, security B, which returned 15.0% and security C, which returned –5.0%. At the beginning of the year 45% was invested in security A, 25.0% in security B and the remaining 30% was invested in security C. Calculate the return of Oliver's portfolio over the year.

$$\mathbf{R_p = (.45x12\%)+(.25x15\%)+(.3x(-5\%)) = 7.65\%}$$

Calculating Portfolio Risk

- While there may be different definitions of risk, one widely-used measure is called variance. Variance measures the variability of realized returns around an average level. The larger the variance the higher the risk in the portfolio.

- Variance is dependent on the way in which individual securities interact with each other. This interaction is known as *covariance.* Covariance essentially tells us whether or not two securities returns are correlated. Covariance measures by themselves do not provide an indication of the degree of correlation between two securities. As such, covariance is standardized by dividing covariance by the product of the standard deviation of two individual securities. This standardized measure is called the ***correlation coefficient***.

Example 3:

Oliver's portfolio holds security A, which returned 12.0% and security B, which returned 15.0%. At the beginning of the year 70% was invested in security A and the remaining 30% was invested in security B. Given a standard deviation of 10% for security A, 20% for security B and a correlation coefficient of 0.5 between the two securities, calculate the portfolio variance.

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A \sigma_B \rho_{AB} \tag{1}$$

Portfolio Variance = $(.7^2 x 10^2)+(.3^2 x 20^2)+(2x.7x.3x10x20x.5) = 127$

Portfolio standard deviation is the square root of the portfolio variance.

$$\sigma_p = \sqrt{127} = 11.27\% \tag{2}$$

Equivalently:

Portfolio Standard Deviation = $(127)^{.5}$=11.27%

The above example is for a portfolio that consists of two assets. You will find an example of a three asset portfolio in the practice questions at the end of this chapter.

Correlation Coefficients

$$\rho_{AB} = \frac{Cov_{AB}}{\sigma_A \sigma_B} \quad \text{or} \quad \rho_{AB} = \frac{.01_{AB}}{10\%_A 20\%_B} = .5 \tag{3}$$

Correlation is the covariance of security A and B divided by the product of the standard deviation of these two securities. It is a pure measure of the co-movement between the two securities and is bounded by –1 and +1.

- A correlation of +1 means that the returns of the two securities always move in the same direction; they are perfectly positively correlated.

- A correlation of zero means the two securities are uncorrelated and have no relationship to each other.

- A correlation of –1 means the returns always move in the opposite direction and are negatively correlated.

Portfolio risk can be effectively diversified (reduced) by combining securities with returns that do not move in tandem with each other.

Example 4:

What happens to the portfolio standard deviation (risk) when the two securities are negatively correlated rather than positively correlated? Using the same data as in Example 3 but now with negative correlation equal to -.5:

$$\textbf{Portfolio Variance} = (.7^2 \text{x} 10^2)+(.3^2 \text{x} 20^2)+(2\text{x}.7\text{x}.3\text{x}10\text{x}20\text{x}(-.5)) = 43$$

$$\textbf{Portfolio Standard Deviation} = (43)^{.5} = 6.55\%$$

The portfolio's risk is reduced from 11.27% to 6.55% when securities that are negatively correlated are combined.

The Markowitz Model

- The Markowitz model describes a set of rigorous statistical procedures used to select the optimal portfolio for wealth maximizing/risk-averse investors.

- The model is described under the framework of a risk-return tradeoff graph.

- Investors are assumed to seek out maximizing their returns while minimizing the risk involved in generating those returns.

- The model recognizes that different investors have different risk tolerance and return requirements. As such, the model defines investor utility functions that can be used to graph indifference curves. These indifference curves plot the various risk levels that give an investor the same level of satisfaction (utility) on a risk-return trade-off graph.

- The model then defines a universe of potential risky securities for investment. Each risky security in the universe requires an estimate of:
  - expected returns,
  - standard deviation,
  - correlation between each pair of risky securities.

- When these sets of securities are combined into a portfolio in such a way as to minimize risk for any desired level of return (or alternatively, maximize returns for any level of risk) then the portfolio represents what is known as the *Markowitz efficient frontier*.

- As you will see in your *Portfolio Management Techniques* CSI Textbook in Figure 4.1 on page 65; the minimum variance portfolio is the left-most point on the graph and the location of the least risky portfolio available from the sets of risky securities.

- The final step in portfolio construction is to combine an investor's utility function (that is, their indifference curves) with the efficient frontier graph. The optimal portfolio for the investor is the indifference curve that is tangent to the efficient frontier.

Risk Free Assets

- The introduction of a risk free security to the Markowitz model changes the efficient frontier from a curved line to a straight line called the Capital Market Line (CML). This CML represents the allocation of capital between risk free securities and risky securities for all investors combined.

- The optimal portfolio for an investor is the point where the new CML is tangent to the old efficient frontier when only risky securities were graphed. This optimal portfolio is normally known as the market portfolio.

Capital Asset Pricing Model (CAPM)

- The CAPM predicts the expected return of a security given:
  - the expected return on the market,
  - the security's beta, and
  - the risk free rate.

- The CAPM model uses the Security Market Line (SML), which is a tradeoff between expected return and security's risk (beta risk) relative to the market portfolio.

- The introduction of a risk free asset changes the risk of a portfolio because risk-free securities have zero covariance and zero correlation with risky assets. Risk is normally reduced when risk-free securities are introduced. It is directly related to the weight allocation in risky assets and inversely related to the weight allocation in risk-free assets.

- The introduction of a risk free security also produces the CML and the market portfolio. Remember that the market portfolio is the sum of the weights of every risky asset in the market.

- Each investor will allocate to the market portfolio and the risk free asset in accordance with their own risk tolerance. All investors, however, are assumed to be rational and therefore will all hold the identical market portfolio, but in different weights. The CAPM model also implies that no other risky portfolio other than the market portfolio will produce greater utility.

Systematic Risk vs. Non-Systematic Risk

- Non-systematic risk is the risk that disappears in the portfolio construction process when you diversify among assets that are not correlated.

- Systematic risk is the risk that remains after constructing the market portfolio, which presumably contains all risky assets. It is the risks that cannot be diversified away.

- Total Risk = Systematic Risk + Non-Systematic Risk.

CAPM Calculation

- The CAPM predicts the expected return of a security given the expected return on the market, the security's beta, and the risk free rate.

Example 5:

The risk-free rate is 7% and the expected return on the market is 15%. If Oliver Company has a beta of 1.2 calculate the expected return on Oliver Company's stock.

$$E(r) = R_f + \beta(R_M - R_f) \tag{4}$$

E(r) = .07+1.2(.15-.07)

E (r) = .166 or 16.6%

Portfolio Beta Calculation

- A portfolio's beta is the weighted average of the individual betas of the securities in the portfolio.

Example 6:

Suppose a portfolio contains three securities with weights of 50%, 25% and 25% respectively. The beta of security A is 1.25. Security B's beta is 0.95 and security C's beta is 1.05. Calculate the beta of the portfolio.

$$\text{Portfolio Beta} = (0.5*1.25) + (0.25*0.95) + (0.25*1.05) = 1.125$$

CAPM Critiques

- Relaxing some of the CAPM assumptions and testing them empirically changes the model's implications.

- Relaxing the assumptions of each investor having the same return expectations, no transaction costs and no taxes will result in multiple Security Market Lines (SMLs) rather than one SML.

- A security's beta changes over time and can be unstable. This means that historical betas cannot be used as proxy for future betas.

- Portfolio betas tend to be more stable than individual security betas.

- Richard Rolls critique of CAPM focuses on the benchmark error problem. It posits that calculated betas are wrong because they may be based on an inappropriate benchmark. The SML may, therefore, be incorrectly specified because it will intersect an improperly specified benchmark portfolio.

- Essentially, Roll's critique of the CAPM focuses on the fact that market benchmarks such as the S&P/TSX Composite Index or the S&P 500 Index do not include all assets, such as human capital, and therefore we cannot observe the return on the true market portfolio.

Market Models vs. Factor Models

- The CAPM is a market model because a security's return is estimated based on its sensitivity to excess returns on the market portfolio. Essentially, the market is the only factor that matters.

- Factor models extend the CAPM to consider that a security's return also depends on other economic measures and not just the market portfolio.

- One multi-factor model is the Arbitrage Pricing Theory Model (APT).

- The APT model predicts a security's return based on unexpected changes in economic factors such as inflation, industrial production, bond risk premium, and the term structure of interest rates. Different factors may be relevant depending on the security under analysis.

- Essentially, factor models do not have all the assumptions of market models such as CAPM.

Questions:

1. Oliver's portfolio holds security A, which returned 12.0%, security B, which returned 15.0% and security C, which returned –5.0%. At the beginning of the year 45% was invested in security A, 25.0% in security B and the remaining 30% was invested in security C. The correlation between AB is 0.75, between AC 0.35, and between BC –0.5. Securities As standard deviation is 12%, security B's standard deviations is 15% and security C's is 10%. Calculate the expected return of Oliver's Portfolio, the portfolio variance and standard deviation.

2. Explain what happens to a portfolio's overall risk when securities that are uncorrelated are combined.

3. List the steps that go into selecting an optimal portfolio of risky assets.

4. Define systematic risk and non-systematic risk.

5. Big Boy Ltd. has a beta of 1.25, the risk free rate is 5% and the excess return on the market is 6%. Calculate the expected return on Big Boy Ltd using CAPM.

6. Name and describe one type of multi-factor model.

Solutions

1. Expected Return

R_p = (.45x12%)+(.25x15%)+(.3x(-5%)) = 7.65%

Portfolio Variance (PV)

$$\sigma_p^2 = [w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2] + [2w_A w_B \sigma_A \sigma_B \rho_{AB} + 2w_A w_C \sigma_A \sigma_C \rho_{AC} + 2w_B w_C \sigma_B \sigma_C \rho_{BC}]$$

PV=[($.45^2$x$.12^2$)+($.25^2$x$.15^2$)+($.30^2$x$.10^2$)]+[(2x.45x.25x.12x.15x0.75)+(2x.45x.3x .12x.10x0.35)+(2x.25x.30x.15x.10x (-0.5))] = .00827 or 0.827%

Portfolio Standard Deviation = $(.827)^{0.5}$ =.909%

2. The risk or standard deviation of a portfolio falls when securities that are uncorrelated are combined. The lower the correlation the greater the diversification benefits as quantified through risk reduction.

3. First, define investor utility by plotting their indifference curve on a risk/return tradeoff graph. Second, define a universe of potential risky securities to be invested in. Third, combine these securities into portfolios that maximize returns and minimize risks. A line connecting each of these portfolios is the efficient frontier. Finally, select the minimum variance portfolio.

4. Systematic risk is risk that cannot be diversified away by combining securities together. It is the risk that remains after diversification. Non-systematic risk is the risk that is diversified away by combining uncorrelated securities together.

5. Expected Return = .05+1.25(.06) = 12.5%

6. One type of multi-factor model is the Arbitrage Pricing Theory Model (APT). APT recognizes that stock returns are affected by several major macroeconomic factors:

   1. Expected returns,
   2. Unexpected changes in systematic factors and
   3. Firm specific factors or diversifiable risk.